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Spinor analysis for the quantum group $SU_q(2)$

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Abstract. Based on the definitions of quantum group $SU_q(2)$ and the two-dimansional quantum plane *à la* Woronowicz and Manin, the covariant spinor calculus, very similar to the classical SU(2) group, is presented. New features arise from the non-commutation among the entries of the quantum matrix and among the 'coordinates' of the quantum plane. q-deformed Pauli matrices are defined and their applications are illustrated.

1. Introduction

Recently, much attention has been paid to the quantum groups. The mathematical structure of the quantum-enveloping algebras has been systematically carried out by Drinfeld [1], Jimbo [2] and by Reshetikhin and co-workers [3]. Another approach to the same subject, mainly in dealing with the quantum formal group itself as a non-commuting space and the so-called quantum plane on which the quantum group acts, was developed by Woronowicz [4] and Manin [5]. In this paper, starting from the quantum plane we develop the covariant spinor calculus method, parallel to that used in the classical SU(2) group, which we believe is more familiar to physicists.

In section 2, we review the spinor calculus in the classical SU(2) group case to fix the notation and conventions. The main tools for the modified spinor calculus, e.g. the projection operators and q-deformed Pauli matrices etc. are examined in section 3, and their applications are outlined in section 4.

A simple spinor calculation has been proposed in [6] and the present work can be considered as a further step in developing and completing this method. Some of the points in this paper have already been outlined in a talk given by Wess [7] and reported in [8]. We include them here for completeness.

2. Gist of SU(2) spinor calculus

The SU(2) group is usually considered as the transformation group acting upon the 2D complex space, (if not otherwise specified, summation over repeated indices is understood)

$$u^\alpha \longrightarrow u'^\alpha = M^\alpha{}_\beta u^\beta \tag{2.1}$$

with $u^\alpha = \binom{u}{v}$ the complex coordinates of the 2D vector and

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2}$$

the non-singular matrix satisfying the unitarity condition

$$M^+ M = E \tag{2.3}$$

where E is a unit 2×2 matrix, and the unimodularity condition

$$\det M = ad - bc = 1\,. \tag{2.4}$$

This gives the following relations among the entries of M:

$$b = -c^* \qquad d = a^* \qquad a^* a + c^* c = 1 \tag{2.5}$$

and the other form of the unitarity condition

$$M M^+ = E\,.$$

The conjugate complex 2D vector, usually represented by a row vector $\bar{u}_\alpha = (u^*, v^*)$, is transformed as

$$\bar{u}_\alpha \longrightarrow \bar{u}'_\alpha \longrightarrow \bar{u}_\beta M^{+\beta}{}_\alpha\,. \tag{2.6}$$

Then the unitarity condition implies that the length of the vector u is invariant,

$$\bar{u}_\alpha u^\alpha = u^* u + v^* v = \mathrm{inv} = u'^* u' + v'^* v' \tag{2.7}$$

while the unimodularity condition gives

$$\epsilon_{\alpha\beta} u'^\alpha u'^\beta = \epsilon_{\alpha\beta} u^\alpha u^\beta = 0 \tag{2.8}$$

where the second equation comes from the commutation of the coordinates and the Levi–Civita symbol $\epsilon_{\alpha\beta}$ and its inverse $\epsilon^{\alpha\beta}$ are given by

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \epsilon \qquad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{-1} \tag{2.9}$$

with

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma\,. \tag{2.10}$$

As a matter of fact $\epsilon^{\alpha\beta}(\epsilon_{\alpha\beta})$ is the eigenvector of $M \otimes M$ with $\det M$ being the associated eigenvalue

$$M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} \epsilon^{\alpha'\beta'} = (\det M)\epsilon^{\alpha\beta} \qquad \epsilon_{\alpha\beta} M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} = \epsilon_{\alpha'\beta'}(\det M)\,. \tag{2.11}$$

For the unimodular matrix, $\det M = 1$, $\epsilon^{\alpha\beta}(\epsilon_{\alpha\beta})$ is an invariant tensor, and then

$$\epsilon_{\gamma\delta} M^{\gamma}{}_{\beta}\epsilon^{\alpha\beta} = M^{-1\alpha}{}_{\delta} \qquad \text{or} \qquad \epsilon^{-1}M^{t}\epsilon = M^{-1} \tag{2.12a}$$

with M^t being the transposition of the matrix M.

For the SU(2) matrix, direct calculation gives

$$\epsilon_{\delta\gamma} M^{+\gamma}{}_{\beta}\epsilon^{\beta\alpha} = M^{\alpha}{}_{\delta} \qquad \text{or} \qquad \epsilon M^{+}\epsilon^{-1} = M^{t}\,. \tag{2.12b}$$

This is indeed the same relation as in (2.12*a*) since $M^{+} = M^{-1}$, and this means that

$$\bar{u}^{\alpha} \equiv \bar{u}_{\beta}\epsilon^{\beta\alpha} \tag{2.13}$$

transforms just as u^α does. This is the feature of SU(2) group—the basic vector u^α and its conjugate $\bar{u}_\alpha = (u^\alpha)^*$ carry the equivalent representation. As is well known, the combination

$$x^{i} \equiv \bar{u}\sigma^{i}u = \bar{u}_{\alpha}\sigma^{i\alpha}{}_{\beta}u^{\beta} \tag{2.14}$$

transform as a triple representation: $x^i \longrightarrow x'^i = D^i{}_j x^j$. Here $D^i{}_j$ is the usual rotation matrix and σ^i are Pauli matrices which can be brought to the normalized canonical form for convenience in the following discussion.

$$\begin{aligned}
\tau_{+} &= \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\tau_{-} &= \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\tau_{3} &= \frac{1}{\sqrt{2}}\sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\tau_{0} &= \frac{1}{\sqrt{2}}E_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,.
\end{aligned} \tag{2.15}$$

The commutation relations now take the form

$$[\tau_3, \tau_{\pm}] = \pm\sqrt{2}\tau_{\pm} \qquad [\tau_{+}, \tau_{-}] = \sqrt{2}\tau_3\,. \tag{2.16}$$

This implies that $(-\tau_{+}\tau_3\tau_{-})$ construct the $j = 1$ irreducible tensor $(\tau_{1,1}, \tau_{1,0}, \tau_{1,-1})$. Later on we denote them as τ_m, $m = (+1), (0), (-1)$ or equivalently $m = +, 3, -$ wherever it is convenient. The dual set of the Pauli matrices are also introduced:

$$\bar{\tau}^{+} = \tau_{-} \qquad \bar{\tau}^{-} = \tau_{+} \qquad \bar{\tau}^{3} = \tau_{3} \qquad \bar{\tau}^{0} = \tau_{0}\,. \tag{2.17}$$

They satisfy the relations

$$\mathrm{Tr}(\bar{\tau}^{\mu}\tau_{\nu}) = (\bar{\tau}^{\mu})^{\alpha}{}_{\beta}(\tau_{\nu})^{\beta}{}_{\alpha} = \delta^{\mu}{}_{\nu} \tag{2.18a}$$

$$(\bar{\tau}^{\mu})^{\alpha}{}_{\beta}(\tau_{\mu})^{\gamma}{}_{\delta} = \delta^{\alpha}{}_{\delta}\delta^{\gamma}{}_{\beta} = E^{\alpha\gamma}{}_{\delta\beta} \tag{2.18b}$$

where $\mu, \nu = +, 3, -, 0$. With the help of the ϵ symbol, we can introduce the antisymmetric tensors

$$s^{\alpha\beta} = \tau_0{}^\alpha{}_\gamma \epsilon^{\gamma\beta} \qquad s_{\alpha\beta} = \epsilon_{\beta\gamma} \bar{\tau}^{0\gamma}{}_\alpha \tag{2.19a}$$

and symmetric tensors

$$t_m{}^{\alpha\beta} = \tau_m{}^\alpha{}_\gamma \epsilon^{\gamma\beta} \qquad t^m{}_{\alpha\beta} = \epsilon_{\beta\gamma} \bar{\tau}^{m\gamma}{}_\alpha \,. \tag{2.19b}$$

The $t^m{}_{\alpha\beta}$ and $t_m{}^{\alpha\beta}$ defined above have the same components and so do $s_{\alpha\beta}$ and $s^{\alpha\beta}$:

$$\begin{aligned} s_{\alpha\beta} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta} \qquad & t^{(+1)}{}_{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \\ t^{(0)}{}_{\alpha\beta} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta} \qquad & t^{(-1)}{}_{\alpha\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta} \,. \end{aligned} \tag{2.20}$$

They satisfy the orthonormality relations

$$s_{\alpha\beta} s^{\alpha\beta} = 1 \qquad t^m{}_{\alpha\beta} t_n{}^{\alpha\beta} = \delta^m{}_n \qquad s_{\alpha\beta} t_m{}^{\alpha\beta} = t^m{}_{\alpha\beta} s^{\alpha\beta} = 0 \,. \tag{2.21}$$

As a matter of fact, $s_{\alpha\beta}$ and $t^m{}_{\alpha\beta}$ are nothing but the C–G coefficients in coupling two doublets into the singlet and triplet respectively. The projection operators corresponding to the singlet and triplet can be easily constructed as

$$Q^{\alpha\beta}{}_{\gamma\delta} = s^{\alpha\beta} s_{\gamma\delta} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\alpha\beta}_{\gamma\delta} \tag{2.22a}$$

$$P^{\alpha\beta}{}_{\gamma\delta} = t_m{}^{\alpha\beta} t^m{}_{\gamma\delta} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}^{\alpha\beta}_{\gamma\delta} \tag{2.22b}$$

with the properties

$$P^2 = P \qquad Q^2 = Q \qquad PQ = QP = 0 \qquad P + Q = E \,. \tag{2.23}$$

The transformation law for the higher rank tensors is given by

$$T^{\alpha\beta\dots}{}_{\gamma\delta\dots} \longrightarrow M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} \dots T^{\alpha'\beta'\dots}{}_{\gamma'\delta'\dots} M^{+\gamma'}{}_\gamma M^{+\delta'}{}_\delta \dots \,. \tag{2.24}$$

3. q-spinors and related topics

As is shown by Manin [5], the quantum group can be considered as transformations acting upon a quantum space, whose coordinates do not commute. Taking the simplest 2D case, the commutation relation between the two 'coordinates' is given by

$$uv = qvu \tag{3.1}$$

with q a complex number. As in the classical case we group u and v into a column vector u^α and call it a q-spinor. The general transformation of a q-spinor is represented by a 2×2 matrix M as in (2.1):

$$u^\alpha \longrightarrow u'^\alpha = M^\alpha{}_\beta u^\beta . \tag{3.2}$$

Introducing the q-deformed Levi–Civita symbol $\epsilon_{\alpha\beta}(q)$ and its inverse $\epsilon^{\alpha\beta}(q)$

$$\epsilon_{\alpha\beta}(q) = \begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta}(q) = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \tag{3.3}$$

we can write the commutation relation (3.1) as

$$\epsilon_{\alpha\beta}(q) u^\alpha u^\beta = 0 . \tag{3.4}$$

Requiring that the transformations preserve the commutation relation (3.1), we find the entries of M are also non-commutative. This requirement demands that $\epsilon_{\alpha\beta}$ is invariant up to a scalar factor,

$$\epsilon_{\alpha\beta}(q) M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} = (\det{}_q M) \epsilon_{\alpha'\beta'}(q) \tag{3.5}$$

which is nothing but the q-analogue of (2.11). As indicated by Woronowicz [4], the twisted unimodularity condition

$$\det{}_q M = 1 \tag{3.6}$$

together with the unitarity conditions (now the following two equations are independent because of the non-commutation among the entries of M)

$$M^+ M = E \tag{3.7a}$$

$$M M^+ = E \tag{3.7b}$$

determine M to be an $SU_q(2)$ matrix for q real. Transformations M do not generate a usual group but the quantum group $SU_q(2)$. (3.6) and (3.7*a*) allow us to write M as

$$M = \begin{pmatrix} a & -q^{1/2} c^* \\ q^{-1/2} c & a^* \end{pmatrix} \tag{3.8}$$

with the entries satisfying the relation

$$a^* a + q^{-1} c^* c = I \qquad a a^* + q c c^* = I \qquad ac = qca \qquad (c^* a^* = q a^* c^*) \tag{3.9}$$

where I is the unit of the algebra A generated by a, c, a^* and c^*. The second unitarity condition (3.7*b*) gives

$$ac^* = qc^*a \qquad cc^* = c^*c\,. \tag{3.10}$$

It has been proved that the algebra A is a Hopf* algebra [4].

Now the transformations of the spinor u^α and its conjugate $\bar{u}_\alpha$ can be written down explicitly as

$$\begin{aligned} u' &= au - q^{1/2}c^*v \qquad & u'^* &= u^*a^* - q^{1/2}v^*c \\ v' &= q^{-1/2}cu + a^*v \qquad & v'^* &= q^{-1/2}u^*c^* + v^*a \end{aligned} \tag{3.11}$$

and then (3.6) gives

$$u'v' - qv'u' = uv - qvu \tag{3.12}$$

and the unitary condition (3.7*a*) implies the invariance of the length

$$\bar{u}_\alpha u^\alpha = u^*u + v^*v = \mathrm{inv}\,. \tag{3.13}$$

But now $u^\alpha \bar{u}_\alpha$ is not invariant. It can be easily seen from (3.11) and (3.9) that

$$q^{-1}u'u'^* + qv'v'^* = q^{-1}uu^* + qvv^* = \mathrm{inv}\,. \tag{3.14}$$

Comparing (3.14) with (3.13) we realize that u^α do not commute with $\bar{u}_\beta$.

It can be shown further that matrix M satisfies the Yang–Baxter relation

$$\check{R}_{12}M_1M_2 = M_1M_2\check{R}_{12} \tag{3.15}$$

where the numerical matrix $\check{R}$

$$(\check{R}) = \check{R}^{\alpha\beta}{}_{\gamma\delta} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{3.16}$$

satisfies the Yang–Baxter equation (in the braid form)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{3.17}$$

and the characteristic equation

$$(\check{R} - q)^3(\check{R} + q^{-1}) = 0\,. \tag{3.18}$$

The eigenvalue equation of $\check{R}$ can be written as

$$\check{R}^{\alpha\beta}{}_{\gamma\delta}t_m(q)^{\gamma\delta} = q\ t_m(q)^{\alpha\beta} \qquad \check{R}^{\alpha\beta}{}_{\gamma\delta}s(q)^{\gamma\delta} = -q^{-1}s(q)^{\alpha\beta}\,. \tag{3.19a}$$

Now since $\check{R}$ is a symmetric matrix, its right-acting eigenvectors (denoted by $t^m(q)_{\alpha\beta}$ and $s(q)_{\alpha\beta}$) have the components identical to those of its left-acting counterparts

$$t^m(q)_{\alpha\beta}\check{R}^{\alpha\beta}{}_{\gamma\delta} = q\ t^m(q)_{\gamma\delta} \qquad s(q)_{\alpha\beta}\check{R}^{\alpha\beta}{}_{\gamma\delta} = -q^{-1}s(q)_{\gamma\delta} \tag{3.19b}$$

$$t^m(q)_{\alpha\beta} = t_m(q)^{\alpha\beta} \qquad s(q)_{\alpha\beta} = s(q)^{\alpha\beta}\,. \tag{3.20}$$

The normalized eigenvectors can now be taken as

$$t^{(+1)}(q)_{11} = 1$$
$$(t^{(0)}(q)_{12}, t^{(0)}(q)_{21}) = (q^{1/2}, q^{-1/2})[2]^{-1/2} \tag{3.21a}$$
$$t^{(-1)}(q)_{22} = 1$$

and

$$(s(q)_{12},\ s(q)_{21}) = (q^{-1/2}, -q^{1/2})[2]^{-1/2} \tag{3.21b}$$

with all the other components being zero. Here the q-number is defined as

$$[n] = \frac{q^{n} - q^{-n}}{q - q^{-1}} \tag{3.22}$$

such that $[0] = 0$, $[1] = 1$, $[-n] = -[n]$, and $[2] = q + q^{-1}$. It is easy to see that $t^m(q)$ and $s(q)$ satisfy the following orthonormality conditions:

$$t_m(q)^{\alpha\beta} t^n(q)_{\alpha\beta} = \delta^n_m \qquad s(q)^{\alpha\beta} s(q)_{\alpha\beta} = 1 \tag{3.23a}$$

$$t_m(q)^{\alpha\beta} s(q)_{\alpha\beta} = s(q)^{\alpha\beta} t^m(q)_{\alpha\beta} = 0 \tag{3.23b}$$

and the symmetric relation

$$t^m(q)_{\alpha\beta} = t^m(q^{-1})_{\beta\alpha} \qquad s(q)_{\alpha\beta} = -s(q^{-1})_{\beta\alpha}\,. \tag{3.24}$$

The projection operators for the triplet and singlet can be defined as

$$\mathcal{P}^{\alpha\beta}{}_{\gamma\delta} = t_m(q)^{\alpha\beta} t^m(q)_{\gamma\delta} \qquad \mathcal{Q}^{\alpha\beta}{}_{\gamma\delta} = s(q)^{\alpha\beta} s(q)_{\gamma\delta} \tag{3.25}$$

respectively, with the same properties as in the classical case, i.e.

$$\mathcal{P}^2 = \mathcal{P} \qquad \mathcal{Q}^2 = \mathcal{Q} \qquad \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0 \qquad \mathcal{P} + \mathcal{Q} = E\,. \tag{3.26}$$

The $\check{R}$ matrix and other relevant matrices can be expressed as the linear combination of $\mathcal{P}$ and $\mathcal{Q}$, e.g.

$$\check{R} \equiv q\mathcal{P} - q^{-1}\mathcal{Q} \equiv \lambda_1 \mathcal{P} + \lambda_0 \mathcal{Q}\,. \tag{3.27}$$

Conversely, the projectors can be re-expressed in terms of $\check{R}$:

$$\mathcal{P} = \frac{\check{R} - \lambda_0 E}{\lambda_1 - \lambda_0} \qquad \mathcal{Q} = \frac{\check{R} - \lambda_1 E}{\lambda_0 - \lambda_1}\,. \tag{3.28}$$

For simplicity we set $t^0(q)_{\alpha\beta} = s(q)_{\alpha\beta}$, put four ts together, and denote them as $t^\mu(q)_{\alpha\beta}$, $\mu = +, 3, -, 0$. The orthonormality conditions now become

$$t_\mu(q)^{\alpha\beta} t^\nu(q)_{\alpha\beta} = \delta_\mu{}^\nu \tag{3.29}$$

and the completeness conditions can be expressed as

$$t_\mu(q)^{\alpha\beta} t^\mu(q)_{\gamma\delta} = E^{\alpha\beta}{}_{\gamma\delta} = \delta^\alpha{}_\gamma \delta^\beta{}_\delta . \tag{3.30}$$

Corresponding to (2.12), direct calculation shows

$$\epsilon(q)_{\alpha\beta} M^{+\beta}{}_\gamma \epsilon(q)^{\gamma\delta} = M^\delta{}_\alpha \qquad (\epsilon(q) M^+ \epsilon^{-1}(q) = M^t) . \tag{3.31}$$

This implies that the conjugate spinor $\bar{u}_\alpha$ transforms equivalent to the basic spinor u^α, i.e.

$$\bar{u}^\alpha = \bar{u}_\beta \epsilon(q)^{\beta\alpha} \tag{3.32a}$$

transforms just as u^α does. Conversely we have

$$\bar{u}_\beta = \bar{u}^\alpha \epsilon(q)_{\alpha\beta} . \tag{3.32b}$$

Consider two different q-spinors u^α and $w^\beta = \binom{w}{z}$ transformed by the same matrix M. Their q-antisymmetric combination is an invariant (singlet)

$$s = s(q)_{\alpha\beta} u^\alpha w^\beta = [2]^{-1/2}(q^{-1/2} uw - q^{1/2} vz) \tag{3.33}$$

and the q-symmetric combination is a triplet

$$t^m = t^m(q)_{\alpha\beta} u^\alpha w^\beta \tag{3.34a}$$

i.e.

$$\begin{aligned} t^+ &= -uw \\ t^3 &= [2]^{-1/2}(q^{1/2} uw + q^{-1/2} vz) \\ t^- &= vz . \end{aligned} \tag{3.34b}$$

Under the $SU_q(2)$ transformation

$$t^m \longrightarrow t'^m = D^m{}_n(q) t^n \tag{3.35}$$

where $D^m{}_n(q) = t^m(q)_{\alpha\beta} M^\alpha{}_\gamma M^\beta{}_\delta t_n(q)^{\gamma\delta}$ is the $j = 1$ representation of $SU_q(2)$, $m, n = +, 3, -$:

$$D^m{}_n = \begin{pmatrix} a^2 & [2]^{1/2} a c^* & -q c^{*2} \\ -[2]^{1/2} ca & 1 - [2] cc^* & -[2]^{1/2} c^* a^* \\ -q^{-1} c^2 & [2]^{1/2} a^* c & a^{*2} \end{pmatrix} . \tag{3.36}$$

By means of the the completeness relation, the product $u^\alpha w^\beta$ can be expressed in terms of t^m and s:

$$\begin{aligned} u^\alpha w^\beta = E^{\alpha\beta}_{\gamma\delta} u^\gamma w^\delta &= (\mathcal{P} + \mathcal{Q})^{\alpha\beta}_{\gamma\delta} u^\gamma w^\delta \\ &= t_m^{\alpha\beta}(t^m_{\gamma\delta} u^\gamma w^\delta) + s^{\alpha\beta}(s_{\gamma\delta} u^\gamma w^\delta) \\ &= t_m(q)^{\alpha\beta} t^m + s(q)^{\alpha\beta} s . \end{aligned} \tag{3.37}$$

Making use of the equivalence relation (3.32), we can complete the reduction for the product of pair of spinors, $\bar{u}_\alpha u^\beta$ or $w^\alpha \bar{w}_\beta$. As in the usual SU(2) case, we must introduce the quantum Pauli matrices. Things become much more complicated because of the non-commutation. Similarly to (2.19) we introduce quantities as follows.

$$\tau_\mu(q)^\alpha{}_\beta = t_\mu(q)^{\alpha\gamma}\epsilon(q)_{\gamma\beta} \qquad \bar{\tau}^\mu(q)^\alpha{}_\beta = \epsilon(q)^{\alpha\gamma}t^\mu(q)_{\beta\gamma} \tag{3.38a}$$

$$\tilde{\tau}_\mu(q)^\alpha{}_\beta = t_\mu(q^{-1})^{\alpha\gamma}\epsilon(q)_{\gamma\beta} \qquad \tilde{\bar{\tau}}^\mu(q)^\alpha{}_\beta = \epsilon(q)^{\alpha\gamma}t^\mu(q^{-1})_{\beta\gamma}\,. \tag{3.38b}$$

It can be proved easily that

$$\mathrm{Tr}(\bar{\tau}^\mu(q)\tau_\nu(q)) = \bar{\tau}^\mu(q)^\alpha{}_\beta\tau_\nu(q)^\beta{}_\alpha = \delta^\mu{}_\nu \tag{3.39a}$$

$$\mathrm{Tr}(\tilde{\bar{\tau}}^\mu(q)\tilde{\tau}_\nu(q)) = \tilde{\bar{\tau}}^\mu(q)^\alpha{}_\beta\tilde{\tau}_\nu(q)^\beta{}_\alpha = \delta^\mu{}_\nu \tag{3.39b}$$

from (3.29) and

$$\tau_\mu(q)^\alpha{}_\beta\bar{\tau}^\mu(q)^\gamma{}_\delta = \delta^\alpha{}_\delta\delta^\gamma{}_\beta = E^{\alpha\gamma}{}_{\delta\beta} \tag{3.40a}$$

$$\tilde{\tau}_\mu(q)^\alpha{}_\beta\tilde{\bar{\tau}}^\mu(q)^\gamma{}_\delta = \delta^\alpha{}_\delta\delta^\gamma{}_\beta = E^{\alpha\gamma}{}_{\delta\beta} \tag{3.40b}$$

from (3.30). Here we list the explicit expression of various τ matrices for later use:

$$\begin{aligned}&\tau_+(q) = \begin{pmatrix}0 & q^{-1/2}\\ 0 & 0\end{pmatrix} \qquad \tau_-(q) = \begin{pmatrix}0 & 0\\ q^{1/2} & 0\end{pmatrix}\\ &\tau_3(q) = \begin{pmatrix}q & 0\\ 0 & -q^{-1}\end{pmatrix}[2]^{-1/2} \qquad \tau_0(q) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}[2]^{-1/2}\end{aligned} \tag{3.41a}$$

$$\begin{aligned}&\bar{\tau}^+(q) = \tau_-(q) \qquad \bar{\tau}^-(q) = \tau_+(q)\\ &\bar{\tau}^3(q) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}[2]^{-1/2} \qquad \bar{\tau}^0(q) = \begin{pmatrix}q^{-1} & 0\\ 0 & q\end{pmatrix}[2]^{-1/2}\end{aligned} \tag{3.41b}$$

$$\begin{aligned}&\tilde{\tau}_+(q) = \tau_+(q) \qquad \tilde{\tau}_-(q) = \tau_-(q)\\ &\tilde{\tau}_3(q) = \bar{\tau}^3(q^{-1}) \qquad \tilde{\tau}_0(q) = \bar{\tau}^0(q^{-1})\end{aligned} \tag{3.41c}$$

$$\begin{aligned}&\tilde{\bar{\tau}}^+(q) = \tau_-(q) \qquad \tilde{\bar{\tau}}^-(q) = \tau_-(q)\\ &\tilde{\bar{\tau}}^3(q) = \tau_3(q^{-1}) \qquad \tilde{\bar{\tau}}^0(q) = \tau_0(q^{-1})\,.\end{aligned} \tag{3.41d}$$

The commutation relations among τ's can be obtained directly by these expressions, i.e.

$$\begin{aligned}&[\tau_3(q), \tau_\pm(q)] = \pm[2]^{1/2}\tau_\pm(q)\\ &q\tau_+(q)\tau_-(q) - q^{-1}\tau_-(q)\tau_+(q) = [2]^{1/2}\tau_3(q)\end{aligned} \tag{3.42}$$

which can be regarded as the q-deformation version of the classical SU(2) algebra (2.16). In comparison with the commutation relation of the generators obtained by Woronowicz [4] from a consistent differential calculus on the non-commutative space of the quantum group,

$$\begin{aligned}&q^2\nabla_1\nabla_0 - q^{-2}\nabla_0\nabla_1 = (1+q^2)\nabla_0\\ &q^2\nabla_2\nabla_1 - q^{-2}\nabla_1\nabla_2 = (1+q^2)\nabla_2\\ &q\nabla_2\nabla_0 - q^{-1}\nabla_0\nabla_2 = \nabla_1\end{aligned} \tag{3.43}$$

we have to make the following identification:

$$\nabla_1 = q[2]^{1/2}\bar{\tau}^3(q) \qquad \nabla_2 = -q^{1/2}\bar{\tau}^+(q) \qquad \nabla_0 = q^{1/2}\bar{\tau}^-(q)\,. \tag{3.44}$$

4. Application

In this section, with the help of the tools developed in the last section, we will illustrate how to reduce the high-rank 'tensors' into irreducible pieces. First consider the product of a pair of conjugate spinors $\bar{u}_\beta u^\alpha$. From (3.40*b*) we have

$$\bar{u}_\beta u^\alpha = (\delta^\alpha{}_\delta \delta^\gamma{}_\beta)\bar{u}_\gamma u^\delta = \tilde{\tau}_\mu(q)^\alpha{}_\beta A^\mu \tag{4.1}$$

with

$$A^\mu = \bar{u}_\gamma \tilde{\bar{\tau}}^\mu(q)^\gamma{}_\delta u^\delta \,. \tag{4.2}$$

Under the action of $SU_q(2)$, we see

$$A^0 = \bar{u}_\beta \tilde{\bar{\tau}}^0(q)^\beta{}_\alpha u^\alpha = [2]^{-1/2}\bar{u}_\alpha u^\alpha \longrightarrow A^0 \,. \tag{4.3}$$

As has been mentioned in the last section, the contraction of $\bar{u}_\beta u^\alpha$ is an invariant, and

$$A^m = \bar{u}_\beta \tilde{\bar{\tau}}^m(q)^\beta{}_\alpha u^\alpha \longrightarrow \bar{u}_{\beta'} M^{+\beta'}{}_\beta \tilde{\bar{\tau}}^m(q)^\beta{}_\alpha M^\alpha{}_{\alpha'} u^{\alpha'} = D^m{}_n(q) A^n \tag{4.4}$$

since

$$M^{+\beta'}{}_\beta \tilde{\bar{\tau}}^m(q)^\beta{}_\alpha M^\alpha{}_{\alpha'} = D^m{}_n(q) \tilde{\bar{\tau}}^n(q)^{\beta'}{}_{\alpha'} \tag{4.5}$$

as can be seen from the appendix.

Similarly from (3.40*a*) we see

$$u^\alpha \bar{u}_\beta = (\delta^\alpha{}_\delta \delta^\gamma{}_\beta) u^\delta \bar{u}_\gamma = \tau_\mu(q)^\alpha{}_\beta \bar{\tau}^\mu(q)^\gamma{}_\delta u^\delta \bar{u}_\gamma = \tau_\mu(q)^\alpha{}_\beta B^\mu \,. \tag{4.6}$$

Under the $SU_q(2)$ transformation

$$u^\alpha \bar{u}_\beta \longrightarrow M^\alpha{}_{\alpha'} u^{\alpha'} \bar{u}_{\beta'} M^{+\beta'}{}_\beta = M^\alpha{}_{\alpha'} \tau_\nu(q)^{\alpha'}{}_{\beta'} M^{+\beta'}{}_\beta B^\nu \,. \tag{4.7}$$

We will see in the appendix that

$$M^\alpha{}_{\alpha'} \tau_0(q)^{\alpha'}{}_{\beta'} M^{+\beta'}{}_\beta = \tau_0(q)^\alpha{}_\beta \tag{4.8a}$$

$$M^\alpha{}_{\alpha'} \tau_n(q)^{\alpha'}{}_{\beta'} M^{+\beta'}{}_\beta = \tau_m(q)^\alpha{}_\beta D^m{}_n(q) \,. \tag{4.8b}$$

Then (4.7) gives

$$\tau_m(q)^\alpha{}_\beta B^m + \tau_0(q)^\alpha{}_\beta B^0 \longrightarrow \tau_m(q)^\alpha{}_\beta D^m{}_n(q) B^n + \tau_0(q)^\alpha{}_\beta B^0 \,. \tag{4.9}$$

This implies that

$$B^0 = \bar{\tau}^0(q)^\beta{}_\alpha u^\alpha \bar{u}_\beta = [2]^{-1/2}(q^{-1}uu^* + qvv^*) \tag{4.10}$$

is an invariant (which coincides with the result in (3.14)), while

$$B^m = \bar{\tau}^m(q)^\beta{}_\alpha u^\alpha \bar{u}_\beta \tag{4.11}$$

transform as the $j = 1$ representation of $SU_q(2)$, i.e.

$$B^m \longrightarrow D^m{}_n(q) B^n \,. \tag{4.12}$$

From the decomposition in (3.34), (3.37), (4.1) and (4.6), we conclude that spinors transformed by the same matrix M (e.g. u and w in (3.37)) or by the relevant matrices (e.g. u and $\bar{u}$ in (4.1)) cannot be commuted. The commutation relation between two different spinors must preserve the singlet–triplet structure. So if

$$u^\alpha w^\beta = K^{\alpha\beta}{}_{\gamma\delta} w^\gamma u^\delta \tag{4.13}$$

we must have the form

$$K = k_1 \mathcal{P} + k_0 \mathcal{Q} \,. \tag{4.14}$$

The consistency of triple products such as uww or uuw constrains K to be $K \propto \check{R}$ for $k_0/k_1 = \lambda_0/\lambda_1 = -q^{-2}$ or $K \propto \check{R}^{-1}$ for $k_0/k_1 = \lambda_0/\lambda_1 = -q^2$.

To illustrate the reduction method for the higher-rank 'tensor', we consider a third-rank 'tensor' with mixed indices, $T^\alpha{}_\beta{}^\gamma$. Care must be taken in dealing with these indices. Their position and order are both important. When one wants to change the order, one must introduce some K matrix as in (4.13). When one wants to raise or lower the index, one uses $\epsilon(q)_{\alpha\beta}$ or $\epsilon(q)^{\alpha\beta}$, e.g. $T^\alpha{}_\beta{}^\gamma \epsilon(q)^{\beta\delta} = T^{\alpha\delta\gamma}$. Pairs of upper and lower indices in the neighbourhood can be contracted. (β, γ) indices can be contracted directly (called A-type contraction) to obtain a 'tensor' two ranks lower, i.e. $R^\alpha = T^\alpha{}_\beta{}^\gamma (\tilde{\bar{\tau}}^0)^\beta{}_\gamma$. (α, β) can also be contracted by taking the trace with the matrix $\bar{\tau}^0(q)$ (called B-type contraction) also obtaining a 'tensor' two ranks lower, i.e. $S^\gamma = \bar{\tau}^0(q)^\beta{}_\alpha T^\alpha{}_\beta{}^\gamma$. The irreducible 'tensors', which are both A-type traceless in (β, γ) and B-type traceless in (α, β), can be obtained by a tedious calculation

$$\hat{T}^\alpha{}_\beta{}^\gamma = T^\alpha{}_\beta{}^\gamma - \frac{[2]}{[3]}\{([2]R^\alpha - S^\alpha)\bar{\tau}_0(q)^\gamma{}_\beta + \tau_0(q)^\alpha{}_\beta([2]S^\gamma - R^\gamma)\} \tag{4.15}$$

comparable with the classical result

$$\hat{T}^\alpha{}_\beta{}^\gamma = T^\alpha{}_\beta{}^\gamma - \tfrac{2}{3}(T^\alpha{}_\sigma{}^\sigma - \tfrac{1}{2}T^\sigma{}_\sigma{}^\alpha)\delta_\beta{}^\gamma - \tfrac{2}{3}\delta^\alpha{}_\beta(T^\sigma{}_\sigma{}^\gamma - \tfrac{1}{2}T^\gamma{}_\sigma{}^\sigma) \,. \tag{4.16}$$

Then by raising the index β, $\hat{T}^{\alpha\beta\gamma}$ are q-symmetric with respect to the transposition of (α, β) and symmetric with respect to (β, γ), and so it is totally q-symmetric to all three indices:

$$\hat{T}^{\alpha\beta\gamma} = q^{(\beta-\alpha)}\hat{T}^{\beta\alpha\gamma} = q^{(\gamma-\beta)}\hat{T}^{\alpha\gamma\beta} = q^{2(\gamma-\alpha)}\hat{T}^{\gamma\beta\alpha} \,. \tag{4.17}$$

In a similar way any high-rank 'tensor' can be reduced by applying the ϵ symbol and quantum Pauli matrices step by step. The irreducible 'tensors' can always be represented by the ones with only q-symmetric upper indices, very similarly to the classical SU(2) case.

Quantum Pauli matrices can also be used in a coupling theory which is invariant under the action of quantum group. For example, when $\psi = \psi^\alpha$ is the basic spinor, the expression

$$\bar{\psi}\Phi_m \tau^m \psi \tag{4.18}$$

is an $SU_q(2)$-invariant coupling, provided Φ^m is transformed according to the $j = 1$ representation

$$\Phi^m \longrightarrow D^m{}_n(q)\Phi^m \tag{4.19}$$

where $\Phi_m = g_{mn}\Phi^n$, with g_{mn} the metric in the three-dimensional space, as defined in the Appendix. The same method can be applied to the quantum $SL_q(2, C)$, i.e. the quantum Lorentz group. The result will be given in a separate paper.

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Appendix

In this appendix we give some useful properties of the projection operators and of the $SO_{q^2}(3)$ transformation matrix $D(M)^m{}_n$.

According to [3], the operator-valued matrix $M = (M^\alpha{}_\beta)_{\alpha,\beta=1,2}$ acting on a linear space V satisfies the Y–B relation in its original form

$$R_{12}M_1M_2 = M_2M_1R_{12} \tag{A1}$$

where $R = R^{\alpha\beta}{}_{\gamma\delta}$ is a numerical matrix associated with $V \otimes V$. Then the compatibility condition for this Y–B relation can be written (sufficiently) as the Y–B equation in its original form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{A2}$$

By introducing the braid-like matrix $\check{R} = PR$ with P the permutation matrix

$$P = P^{\alpha\beta}{}_{\gamma\delta} = E^{\alpha\beta}{}_{\delta\gamma} = \delta^\alpha{}_\delta\delta^\beta{}_\gamma \tag{A3}$$

one can recast the Y–B relation (A1) into its braid form as in (3.15)

$$\check{R}_{12}M_1M_2 = M_1M_2\check{R}_{12} \tag{A4}$$

and (A2) into

$$\check{R}_{12}R_{13}R_{23} = R_{13}R_{23}\check{R}_{12} \tag{A5}$$

simply by multiplying the permutation matrix P_{12} from the left to (A1) and (A2). Further multiplying $P_{23}P_{13}$ on (A5) from the left, one finds the braid form of the Y–B equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{A6}$$

as is given in (3.17). The formal similarity between (A1) and (A2) and between (A4) and (A5) indicates that R can be considered as a simplest representation of M, i.e.

$$(M^{\alpha}{}_{\beta})^{\gamma}{}_{\delta} = kR^{\alpha\gamma}{}_{\beta\delta} \tag{A7}$$

where k must take to be $q^{-1/2}$ to ensure $\det_q M = 1$.

Now the matrix $\check{R}$ has two different eigenvalues namely, $\lambda_1 = q$ (triple) and $\lambda_0 = -q^{-1}$ (single), with $t_m(q)^{\alpha\beta}$ and $s(q)^{\alpha\beta}$ as its corresponding left-action eigenvectors and $t^m(q)_{\alpha\beta}$ and $s(q)_{\alpha\beta}$ as its right-action eigenvectors. The projection operators for the triplet and singlet can be respectively defined as ($Q^{(1)} = \mathcal{P}$, $Q^{(0)} = \mathcal{Q}$ in the text)

$$Q^{(1)\ \alpha\beta}{}_{\gamma\delta} = t_m(q)^{\alpha\beta}t^m(q)_{\gamma\delta} \qquad Q^{(0)\ \alpha\beta}{}_{\gamma\delta} = s(q)^{\alpha\beta}s(q)_{\gamma\delta} \tag{A8}$$

with the properties

$$\begin{aligned} &Q^{(1)\ \alpha\beta}{}_{\gamma\delta}t_m(q)^{\gamma\delta} = t_m(q)^{\alpha\beta} \qquad Q^{(0)\ \alpha\beta}{}_{\gamma\delta}t_m(q)^{\gamma\delta} = 0 \\ &Q^{(0)\ \alpha\beta}{}_{\gamma\delta}s(q)^{\gamma\delta} = s(q)^{\alpha\beta} \qquad Q^{(1)\ \alpha\beta}{}_{\gamma\delta}s(q)^{\gamma\delta} = 0\,. \end{aligned} \tag{A9}$$

And similarly for the right-action vectors $t^m(q)_{\alpha\beta}$ and $s(q)_{\alpha\beta}$. Alternatively the projection operators can be re-expressed by $\check{R}$ itself as in (3.38)

$$Q^{(1)} = \frac{\check{R} - \lambda_0 E}{\lambda_1 - \lambda_0} \qquad Q^{(0)} = \frac{\check{R} - \lambda_1 E}{\lambda_0 - \lambda_1} \tag{A10}$$

with E the unit matrix in $V \otimes V$. Then from (A4), (A5) and (A10) one can easily obtain

$$Q^{(i)}_{12}M_1M_2 = M_1M_2Q^{(i)}_{12} \qquad i = 0,1 \tag{A11a}$$

$$Q^{(i)}_{12}R_{13}R_{23} = R_{13}R_{23}Q^{(i)}_{12} \qquad i = 0,1\,. \tag{A11b}$$

Multiplying s_{12} ($= s(q)_{\alpha\beta}$) from the left or s^{12} ($= s(q)^{\alpha\beta}$) from the right, one gets

$$\begin{aligned} &s_{12}M_1M_2 = s_{12}M_1M_2Q^{(0)}_{12} \\ &Q^{(0)}_{12}M_1M_2s^{12} = M_1M_2s^{12} \end{aligned} \tag{A12a}$$

$$\begin{aligned} &s_{12}R_{13}R_{23} = s_{12}R_{13}R_{23}Q^{(0)}_{12} \\ &Q^{(0)}_{12}R_{13}R_{23}s^{12} = R_{13}R_{23}s^{12}\,. \end{aligned} \tag{A12b}$$

Equation (A12a) shows that $s_{12}M_1M_2$ and $M_1M_2s^{12}$ are the eigenvectors of $Q^{(0)}_{12}$ acting from right and left respectively. So they must be proportional to s_{12} and s^{12}, i.e.

$$\begin{aligned} &s_{12}M_1M_2 = \lambda(M)s_{12} \\ &M_1M_2s^{12} = \mu(M)s^{12} \end{aligned} \tag{A13a}$$

where λ and μ are the proportional coefficients, may depend on the entries of M. But since M under consideration is a unimodular matrix, according to (3.5), one has $\lambda(M) = \mu(M) = 1$. Similarly from (A12*b*) one sees

$$s_{12}R_{13}R_{23} = \lambda'(M)E_3 s_{12}$$
$$R_{13}R_{23}s^{12} = \mu'(M)E_3 s^{12} \qquad \text{(A13}b\text{)}$$

and $\lambda'(M) = \mu'(M) = q$ from the consideration that R is a representation of M as in (A7). This gives

$$s_{12}R_{13} = q s_{12}R_{23}^{-1}$$
$$R_{23}s^{12} = qR_{13}^{-1}s^{12} \qquad \text{(A14}a\text{)}$$

or explicitly

$$s_{\alpha\beta}R^{\alpha\gamma}{}_{\alpha'\gamma'} = q s_{\alpha'\beta'}R^{-1\ \beta'\gamma}{}_{\beta\gamma'}$$
$$R^{\beta\gamma}{}_{\beta'\gamma'}s^{\alpha\beta'} = qR^{-1\ \alpha\gamma}{}_{\alpha'\gamma'}s^{\alpha'\beta} \qquad \text{(A14}b\text{)}$$

or equivalently

$$s_{\alpha\beta}\check{R}^{\gamma\alpha}{}_{\alpha'\gamma'} = q s_{\alpha'\beta'}\check{R}^{-1\ \beta'\gamma}{}_{\gamma'\beta}$$
$$\check{R}^{\gamma\beta}{}_{\beta'\gamma'}s^{\alpha\beta'} = q\check{R}^{-1\ \alpha\gamma}{}_{\gamma'\alpha'}s^{\alpha'\beta}\,. \qquad \text{(A15}a\text{)}$$

Similar relations

$$s_{\alpha\beta}\check{R}^{-1\ \gamma\alpha}{}_{\alpha'\gamma'} = q^{-1}\ s_{\alpha'\beta'}\check{R}^{\beta'\gamma}{}_{\gamma'\beta}$$
$$\check{R}^{-1\ \gamma\beta}{}_{\beta'\gamma'}s^{\alpha\beta'} = q^{-1}\ \check{R}^{\alpha\gamma}{}_{\gamma'\alpha'}s^{\alpha'\beta} \qquad \text{(A15}b\text{)}$$

can be obtained from the consideration that $\check{R}^{-1}$ is another representation of M, i.e. $(M^{\alpha}{}_{\beta})^{\gamma}{}_{\delta} = q^{1/2}\check{R}^{-1\ \gamma\alpha}{}_{\delta\beta}$.

In a similar way, by multiplying t^{12}_m $(= t_m(q)^{\alpha\beta})$ from the right or t^m_{12} $(= t^m(q)_{\alpha\beta})$ from the left to (A11) one obtains

$$t^m_{12}M_1M_2 = t^m_{12}M_1M_2Q^{(1)}_{12}$$
$$Q^{(1)}_{12}M_1M_2t^{12}_m = M_1M_2t^{12}_m \qquad \text{(A16}a\text{)}$$
$$t^m_{12}R_{13}R_{23} = t^m_{12}R_{13}R_{23}Q^{(1)}_{12}$$
$$Q^{(1)}_{12}R_{13}R_{23}t^{12}_m = R_{13}R_{23}t^{12}_m\,. \qquad \text{(A16}b\text{)}$$

Now (A16*a*) shows that $M_1M_2t^{12}_m$ is an eigenvector of $Q^{(1)}_{12}$, so it must be a linear combination of t^{12}_n, i.e.

$$M_1M_2\ t^{12}_m = t^{12}_n\ D(M)^n{}_m \qquad \text{(A17)}$$

where, generally speaking, the combination coefficients $D(M)^n{}_m$ will depend on the entries of M.

Inserting (A8) into (A16*a*), one sees immediately that

$$M_1 M_2 t^{12}_m = t^{12}_n t^n_{12} M_1 M_2 t^{12}_m . \tag{A18}$$

This gives

$$D(M)^n{}_m = t^n{}_{12} M_1 M_2 \; t_m{}^{12} = t^n(q)_{\alpha\beta} M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} t_m(q)^{\alpha'\beta'} . \tag{A19}$$

Similarly

$$t^m{}_{12} M_1 M_2 = t^m{}_{12} M_1 M_2 \; t_n{}^{12} t^n{}_{12} = D(M)^m{}_n t^n{}_{12} . \tag{A20}$$

It is not difficult to show that

$$\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23} M_1 M_2 M_3 M_4 = M_1 M_2 M_3 M_4 \check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23} . \tag{A21}$$

Then one has

$$\check{\mathcal{R}}^{mn}{}_{pq} D^p{}_r D^q{}_s = D^m{}_p D^n{}_q \check{\mathcal{R}}^{pq}{}_{rs} \tag{A22}$$

where

$$\check{\mathcal{R}}^{mn}{}_{pq} = q^{-2} t^m{}_{12} t^n{}_{34} \left(\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}\right) t_p{}^{12} t_q{}^{34} . \tag{A23}$$

Now define the 3D 'coordinates' as

$$\begin{aligned} x^+ &= -t^{(+1)}{}_{\alpha\beta} u^\alpha u^\beta = -u^2 \\ x^0 &= t^{(0)}{}_{\alpha\beta} u^\alpha u^\beta = (q^{1/2} uv + q^{-1/2} vu)[2]^{-1/2} \\ x^- &= t^{(-1)}{}_{\alpha\beta} u^\alpha u^\beta = v^2 . \end{aligned} \tag{A24}$$

Then one sees from (3.35) that under $SU_q(2)$ transformation

$$x^m \longrightarrow D(M)^m{}_n x^n \tag{A25}$$

with $D(M)^m{}_n$, $m, n = +, 0, -$ given in (3.36). Then (A23) gives

$$\check{\mathcal{R}} = \begin{array}{c} \\ ++ \\ +0 \\ +- \\ 0+ \\ 00 \\ 0- \\ -+ \\ -0 \\ -- \end{array} \begin{array}{c} \begin{array}{ccccccccc} ++ & +0 & +- & 0+ & 00 & 0- & -+ & -0 & -- \end{array} \\ \left(\begin{array}{ccccccccc} q^2 & & & & & & & & \\ & \Delta & & 1 & & & & & \\ & & (1-q^{-2})\Delta & & -q^{-1}\Delta & & q^{-2} & & \\ & 1 & & 0 & & & & & \\ & & -q^{-1}\Delta & & 1 & & & & \\ & & & & & \Delta & & 1 & \\ & & q^{-2} & & & & 0 & & \\ & & & & & 1 & & 0 & \\ & & & & & & & & q^2 \end{array}\right) \end{array} \tag{A26}$$

where $\Delta = q^2 - q^{-2}$ and the blank spaces mean that the corresponding entries are zero. Similarly to the two-dimensional case (3.19), there exists a tensor g_{mn} which is the right eigenvector of $\check{\mathcal{R}}^{mn}{}_{m'n'}$,

$$g_{mn}\check{\mathcal{R}}^{mn}{}_{m'n'} = q^{-4} g_{m'n'} \tag{A27}$$

and satisfies the relation

$$g_{mn} D(M)^m{}_{m'} D(M)^n{}_{n'} = g_{m'n'} \tag{A28}$$

g_{mn} and its inverse g^{mn}, playing the role of metric and its inverse, have the same components, i.e.

$$g_{mn} = (g_{+-}, g_{00}, g_{-+}) = (q^{-1}, 1,\ q) \tag{A29}$$

and other components are zero. Comparing to the definition in [3], one sees that $D(M)$ is the operator-valued $\mathrm{SO}_{q^2}(3)$ matrix, and $\mathcal{R}$ is the corresponding R matrix.

Then it can be shown directly from the definition (3.38) that

$$\begin{aligned} M^\alpha{}_\beta \tau_m(q)^\beta{}_\gamma M^{+\gamma}{}_\delta &= M^\alpha{}_\beta t_m(q)^{\beta\sigma} \epsilon(q)_{\sigma\gamma} M^{+\gamma}{}_\delta \\ &= M^\alpha{}_\beta t_m(q)^{\beta\sigma} M^\rho{}_\sigma \epsilon_{\rho\delta} \\ &= t_n(q)^{\alpha\rho} D(M)^n{}_m \epsilon_{\rho\delta} \\ &= \tau_n(q)^\alpha{}_\delta D(M)^n{}_m \end{aligned} \tag{A30}$$

and similarly

$$M^{+\ \alpha}{}_\beta \bar{\bar{\tau}}^m(q)^\beta{}_\gamma M^\gamma{}_\delta = D(M)^m{}_n \bar{\bar{\tau}}^n(q)^\alpha{}_\delta\,. \tag{A31}$$

References

[1] Drinfeld V G 1986 Quantum groups *Proc. ICM* vol 1 p 798

[2] Jimbo M 1986 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247; 1986 *Commun. Math. Phys.* **102** 537

[3] Reshetikhin N Yu 1987 *LOMI preprint* E-4-87; 1987 *LOMI preprint* E-17-87
Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193

[4] Woronowicz S L 1987 *Publ. RIMS (Kyoto University)* vol 23 p 117; 1987 *Commun. Math. Phys.* **111** 613

[5] Manin Yu I 1988 Quantum Groups and Non-commutative Geometry *Les Publications du Centre de Recherches Mathématiques (Université de Montréal)*

[6] Schlieker M and Scholl M 1990 *Z. Phys.* C **47** 625

[7] Wess J 1990 *300-Jahrfeier der Mathematischen Gesellschaft (Hamburg)* oral presentation

[8] Carow-Watamura U, Schlieker M, Scholl M and Watamura S 1990 *Z. Phys.* C **48** 159; 1991 *Int. J. Mod. Phys.* A **6** 3081